BLOCK STRUCTURE OF CERTAIN SERIES OF EGD AND HYPERCUBIC DESIGNS

By

K. R. Aggarwal

Department of Mathematics and Statistics,
Punjab Agricultural University, Ludhiana

(Received : November, 1973)

1. Introduction

The block structure of certain series of group divisible designs was studied by Bose and Connor (1952) and of certain series of triangular and $L_2(s)$ designs was studied by Raghavarao (1960). In this paper, a study of block structure of certain series of extended group divisible (EGD) designs of Hinkelmann and Kempthorne (1963) and hypercubic designs of Shah (1958) and Kusumoto (1965), is made. This study throws light on the possible series of (i) EGD designs which can be taken as the confounded $s_1 \times s_2 \times \ldots \times s_m$ asymmetrical factorial experiments without confounding the main effect of a particular factor, and (ii) hypercubic designs which can be taken as the confounded s^m symmetrical factorial experiments without confounding the main effects of all the factors.

For the definitions of statistical terms used in this paper, we refer to Raghavarao (1971).

2. EGD and Hypercubic Designs

An extended group divisible (EGD) design of Hinkelmann and Kempthorne (1963) is defined as :

Definition 2.1. An EGD design is an arrangement of $v = \prod_{i=1}^{m} (s_i)$ treatments, in b blocks each of size $k(\leq v)$ such that (1) every treatment occurs at most once in a block;

(2) every treatment occurs exactly in r blocks;

(3) let the v treatments be denoted by the elements of the set $[(x_1, x_2, \ldots, x_m): x_i = 1, 2, \ldots, s_i]$. Two treatments $(x_1, x_2, \ldots, x_m)$ and $(x'_1, x'_2, \ldots, x'_m)$ are called $c_1c_2\ldots c_m$th associates, where $c_i = 1$ when $x_i \neq x'_i$, $c_i = o$ when $x_i = x'_i$ $(i = 1, 2, \ldots, m)$; and

(4) pairs of treatments which are $c_1c_2 \ldots c_m$th associates, occur together in $\lambda c_1c_2 \ldots c_m$ blocks.

Let $v=(s_1)(s_2)\ldots(s_m)$, b, r, k, $\lambda c_1c_2 \ldots c_m$'s be the parameters of an EGD design with (2^m-1) associate classes. Let these v treatments be partitioned into s_i disjoint subsets S_j's, each consisting of v/s_i treatments, given as follows :

(2.1) $S_j=\{(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_m) : x_l=1, 2, \ldots, s_l\}, j=1, 2, \ldots, s_i.$

Let

$$a_o^i=r-\lambda c_1c_2 \ldots c_m,\ c_i=1;\ c_j=o,\ j=1, 2, \ldots, m, j\neq i. \tag{2.2}$$

Further, let

$$g_j^i = \sum_{\substack{i_1, i_2, \ldots, i_j=1 \\ i_1<i_2\ldots<i_j}}^{m} 1[\lambda c_1c_2 \ldots c_m\,(s_{i_1}=1)(s_{i_2}=1)\ldots(s_{i_j}=1)],$$

$i_1, i_2, \ldots\ i_j \neq i;\ c_j=1$, when $j=i_1, \ldots, i_j$; $c_j=o$, otherwise

$$h_j^i = \sum_{\substack{i_1, i_2, \ldots, i_j=1 \\ i_1<i_2<\ldots<i_j}}^{m} 1[\lambda c_1c^2 \ldots c_m\,(s_{i_1}-1)(s_{i_2}-1)\ldots(s_{i_j}-1)] \tag{2.3}$$

$i_1, i_2, \ldots\ i_j \neq i;\ c_j=1$, when $j=i_1,\ \ , i_j, i$; $c_j=o$, otherwise

Let

$$a_j^i = g_j^i - h_j^i,\ j=1, 2, \ldots, (m-1). \tag{2.4}$$

Then we shall prove

Theorem 2.1. If in an EGD design

$$a_o^i + a_1^i + \ldots + a_{m-1}^i = o \tag{2.5}$$

and the v treatments of this EGD design, are partitioned into s_i disjoint subsets S_j given by (2.1), then k is divisible by s_i. Further, every block of the design contains k/s_i treatments from each of the subsets $S_1, S_2, \ldots, S_{s_i}$.

Proof : Let the number of treatments that occur in the lth block from the S_j subset be e_l^j. Then we have

$$\sum_l (e_l^j) = vr/s_i,$$

$$\sum_l e_l^j\,(e_l^j-1) = v\left[\ g_1^i + g_2^i + \ldots + g_{m-l}^i\right]/s_i\ . \tag{2.6}$$

Let

$$e^j_{.} = (\sum e^j_l)/b = k/s_i. \quad (2.7)$$

Then

$$\sum (e^j_l - e^j_{.})^2 = v\left[g^i_1 + g^i_2 + \dots + g^i_{m-1} + r\right]/s_i - vrk/s_i^2 \quad (2.8)$$

$$= \left(v/s_i^2\right)\left[s_i r + s_i g^i_1 + \dots + s_i g^i_{m-1} - \left\{ r + (s_i - 1)\left(r - a^i_0\right) + g^i_1 + \dots + g^i_{m-1} + (s_i - l)(h^i_1 + \dots + h^i_{m-1})\right\}\right]$$

$$= \left[mv(s_i - 1)/s_i^2\right]\left[a^i_0 + a^i_1 + \dots + a^i_{m-1}\right] = O.$$

This shows that

(2.9) $$e^j_1 = e^j_2 = \dots = c^j_b = e^j_{.} = k/s_i$$

and since e^j_l's must be positive integers, this implies that k is divisible by s_i. Hence Theorem 2.1 is established.

Illustration 2.1.1. Let $v=(s_1)(s_2)(s_3)$, b, r, k, λ_{100}, λ_{010}, λ_{001}, λ_{110}, λ_{101}, λ_{011}, λ_{111} be the parameters of an EGD design. Then

$$S_j - \{(x_1, x_2, j) : x_i = 1. 2. \dots, s_i\} = j = 1, 2, \dots, s_3.$$

$$a^3_0 = r - \lambda_{001},$$

$$g^3_1 = \lambda_{100}(s_1 - 1) + \lambda_{010}(s_2 - 1),$$

(2.10)

$$g^3_2 = \lambda_{110}(s_1 - 1)(s_2 - 1),$$

$$h^3_1 = \lambda_{101}(s_1 - 1) + \lambda_{011}(s_2 - 1),$$

$$h^3_2 = \lambda_{111}(s_1 - 1)(s_2 - 1),$$

$$a^3_0 + a^3_1 + a^3_2 = r - \lambda_{001} + (\lambda_{100} - \lambda_{101})(s_1 - 1) + (\lambda_{010} - \lambda_{011})(s_2 - 1) + (\lambda_{110} - \lambda_{111})(s_1 - 1)(s_2 - 1).$$

$a^3_0 + a^3_1 + a^3_2 = O$ implies that k is divisible by s_3 and further every block of the design contains k/s_3 treatments from each of the subsets S_1, S_2, ... S_{s_3}.

Let s_i $(i=1, 2, \ldots, m)$, be equal to s. Then the EGD design discussed above becomes the hypercubic design of Shah (1958) and Kusumoto (1965) with the parameters $v=s^m$, b, r, k, $\lambda_1, \ldots, \lambda_m$ and each of the disjoint subsets $S_1, S_2, \ldots, S_s$ will contain s^{m-1} treatments. The following corollary immediately follows from Theorem 2.1 :

Corollary 2.1.1. If in a hypercubic design with the parameters $v=s^m$, b, r, k, $\lambda_1, \ldots, \lambda_m$ the condition

$$(2.11)\quad r+\lambda_1[(s-1)\ (m-1)-1]+(s-1)\lambda_2\left[(s-1)\binom{m-1}{2}-\binom{m-1}{1}\right]+\ldots$$

$$+(s-1)^{m-2}\lambda_{m-1}\left[(s-1)\binom{m-1}{m-1}-\binom{m-1}{m-2}\right]-(s-1)^{m-1}\lambda m=0$$

is satisfied and the s^m treatments are partitioned into $S_1, S_2, \ldots S_s$ disjoint subsets each containing s^{m-1} treatments, as explained earlier, then k is divisible by s. Further, every block of the design will contain k/s treatments from each of the subsets $S_1, S_2, \ldots, S_s$.

Now, we shall discuss the series of EGD designs satisfying the condition as given in (2.5) and of hypercubic designs satisfying the condition as given in (2.11).

Let the $(s_1)\ (s_2) \ldots (s_m)$ treatment combinations of the $s_1 \times s_2 \times \ldots \times s_m$ asymmetrical factorial experiment in factors $F_1, F_2, \ldots, F_m$, F_i being at s_i $(i=1, 2, \ldots, m)$ levels, be denoted by the treatments of the EGD design [See, Aggarwal (1974), p. 316] with the parameters

$$v=(s_1)\ (s_2)\ldots(s_m),\ b=[s_1\ (s_1\text{-}1)]\ldots[s_{m-1}\ (s_{m-1}-1)]$$
$$r=(s_1-1)\ldots(s_{m-1}-1),\ k=s_m,$$

$$(2.12)\quad \lambda c_1 c_2\ldots c_m=1, \text{ when all } c_1\text{'s are unity,}$$
$$=0, \text{ otherwise}$$

where $s_m \leqslant s_1, \ldots s_{m-1}$ and $s_1, \ldots s_{m-i}$ are primes or prime powers.

The condition $\sum_j (a_j^u)=0$ is satisfied by the parameters of the EGD design given in (2.12). But $\sum_j (a_j^m)$ is the latent root of NN′ with multiplicity (s_m-1), where N is the incidence matrix of the EGD design given in (2.12). This implies [See Aggarwal (1974), p. 318] that (s_m-1) orthogonal contrasts of the treatment effects pertaining to the main effect of the factor F_m, are left unconfounded.

Let, in a similar way, the s^m treatment combinations of the s^m symmetrical factorical experiment in m factors $F_1, F_2, \ldots, F_m$, each factor being at s levels, be denoted by the treatments of the hypercubic design [See, Aggarwal (1974), p. 318] with the parameters

$$v=s^m,\ b=[s(s-1)]^{m-1},\ x=(s-1)^{m-1},\ k=s,\quad \lambda_1=\lambda_2=\ldots=\lambda_{m-1}=0,\ \lambda_m=1 \tag{2.13}$$

where s is a prime or a prime power.

The condition $\Sigma_j(a^i_j)=0$ given in (2.11) is satisfied by the parameters of the hypercubic design given in (2.13). Noting that $\Sigma(a^i_j)$ is the latent root of $N^*N^{*\prime}$ with multiplicity $m(s-1)$, where N^* is the incidence matrix of the hypercubic design with the parameters as given in (2.13), we can easily see, as indicated by Aggarwal (1974), p. 318, that the $m(s-1)$ orthogonal contrasts of the treatment effects pertaining to the main effects of all the factors, are left unconfounded.

Summary

This paper contains a study of the block structure of certain series of EGD and hypercubic designs.

Acknowledgement

The author is grateful to Dr. T.A. Ramasubban, Professor and Head, Department of Mathematics and Statistics, Punjab Agricultural Univarsity, for providing the necessary facilities to carry out thiswork.

References

(1) Aggarwal, K.R. : Some higher class PBIB designs and their applications as confounded factoricl experiments. Ann. Inst. State. Math., 1974, 26, 315-323.

[2] Bose, R.C. and Connor, W.S. : Combinatorial properties of group divisible incomplete block designs. Ann. Math. Stat. 1952, 23, 367-383.

[3] Hinkelmann, K. and Kempthorne, O. : Two classes of group divisible partial diallel crosses. Biometrika, 1963, 50, 281-291.

[4] Kusumoto, K. : Hypercubic designs. Wakayama Medical Reports, 1965, 9, 123-132.

[5] Raghavarao, D : On the block structure of certain PBIB designs with triangular and L_2 (s) association schemes. Ann. Math. Stat., 1960, 31, 787-791.

[6] —— : *Constructions and Combinatorial Problems in Design of Experiments.* 1971, Wiley, New York.

[7] Shah, B. V. : On balancing in factorial experiments. Ann. Math. Stat., 1958, 29, 766-779.